# Projectively equivariant symbol calculus

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**Abstract**

The spaces of linear differential operators $\mathcal{D}_\lambda(\mathbb{R}^n)$ acting on $\lambda$-densities on $\mathbb{R}^n$ and the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ of functions on $T^*\mathbb{R}^n$ which are polynomial on the fibers are not isomorphic as modules over the Lie algebra $\mathrm{Vect}(\mathbb{R}^n)$ of vector fields of $\mathbb{R}^n$. However, these modules are isomorphic as $\mathrm{sl}(n+1,\mathbb{R})$-modules where $\mathrm{sl}(n+1,\mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n)$ is the Lie algebra of infinitesimal projective transformations. In addition, such an $\mathrm{sl}_{n+1}$-equivariant bijection is unique (up to normalization). This leads to a notion of projectively equivariant quantization and symbol calculus for a manifold endowed with a (flat) projective structure.

We apply the $\mathrm{sl}_{n+1}$-equivariant symbol map to study the $\mathrm{Vect}(M)$-modules $\mathcal{D}^k_\lambda(M)$ of $k$-th-order linear differential operators acting on $\lambda$-densities, for an arbitrary manifold $M$ and classify the quotient-modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$.

## 1 Introduction

Roughly speaking, a quantization procedure associates linear operators on a Hilbert space to functions on a symplectic manifold. In particular, if this manifold is the cotangent bundle $T^*M$ of a smooth manifold $M$, a now standard quantization procedure leads to a linear bijection from the space $\mathrm{Pol}(T^*M)$ of functions on $T^*M$ which are polynomial on the fibers (or, equivalently, symmetric contravariant tensor fields on $M$), into the space $\mathcal{D}(M)$ of linear differential operators on $M$ :

$$Q : \mathrm{Pol}(T^*M) \to \mathcal{D}(M)\,, \qquad \text{quantization map}$$

The inverse

$$\sigma = Q^{-1} : \mathcal{D}(M) \to \mathrm{Pol}(T^*M)\,, \qquad \text{symbol map}$$

associates to each operator $A \in \mathcal{D}(M)$ a sort of *total* symbol.

**1.1** One of the main questions usually arising in this context is to find the group of symmetries, that is, the Lie group acting on $M$ such that the quantization procedure is *equivariant* with respect to this action. It is natural then to consider the space of differential operators $\mathcal{D}(M)$ as a module over the group $\mathrm{Diff}(M)$ of diffeomorphisms of $M$.

There exists a natural family of $\mathrm{Diff}(M)$- and $\mathrm{Vect}(M)$-module structures on $\mathcal{D}(M)$. To define it, one considers differential operators as acting on tensor-densities of arbitrary

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degree $\lambda$, this leads to a $\mathrm{Diff}(M)$-module of differential operators $\mathcal{D}_\lambda(M)$. The module $\mathcal{D}_{1/2}(M)$ on half-densities plays a special role (see [2, 11]).

It worth noticing that modules of differential operators on tensor densities have been studied in recent papers [6, 13, 9, 4].

**1.2** One of the difficulties of quantization is that there is *no natural quantization map.* In other words, $\mathrm{Pol}(T^*M)$ and $\mathcal{D}_\lambda(M)$ *are not isomorphic* as modules over the group $\mathrm{Diff}(M)$ of diffeomorphisms of $M$ nor as modules over its Lie algebra $\mathrm{Vect}(M)$ of vector fields.

The main idea of this paper is to fix a *maximal group of symmetries* for which such quantization is still possible. We consider a manifold $M$ (of dimension $n$) endowed with a (flat) projective structures (i.e. we locally identify $M$ with the projective space $\mathbb{RP}^n$). We show that there exist *unique* (up to a natural normalization) symbol and quantization maps equivariant with respect to the group $\mathrm{SL}(n+1,\mathbb{R})$ of projective symmetries.

In terms of modules of differential operators, our main result can be formulated as follows. We consider a natural embedding $\mathrm{sl}(n+1,\mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n)$ ($\mathrm{sl}(n+1,\mathbb{R})$ acts on $\mathbb{R}^n$ by infinitesimal projective transformations). It turns out that $\mathcal{D}_\lambda(\mathbb{R}^n)$ *and* $\mathrm{Pol}(T^*\mathbb{R}^n)$ *are equivalent* $\mathrm{sl}(n+1,\mathbb{R})$*-modules.* In particular, the $\mathrm{sl}(n+1,\mathbb{R})$-modules $\mathcal{D}_\lambda(\mathbb{R}^n)$ with different values of $\lambda$ are isomorphic to each other.

**1.3** We apply our $\mathrm{sl}_{n+1}$-equivariant symbol map to the problem of classification of modules of differential operators on an arbitrary smooth manifold $M$. The classification of $\mathrm{Vect}(M)$-modules $\mathcal{D}^k_\lambda(M)$ has been performed in a series of recent works [6, 13] and [9]. In this paper we classify the quotient-modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$. We prove that for $k-\ell \geq 2$ every such module (except the special case $\lambda = 1/2$ with $k-\ell = 2$) is a *nontrivial deformation* of the $\mathrm{Vect}(M)$-module of symbols and compute the corresponding cohomology classes of $\mathrm{Vect}(M)$.

**1.4** An important aspect discussed in this paper is the property of *locality.* We show that any linear map on $\mathrm{Pol}(T^*\mathbb{R}^n)$ is necessarily local if it is equivariant with respect to the Lie group generated by translations and homotheties of $\mathbb{R}^n$ (i.e., with respect to $\mathbb{R}^* \ltimes \mathbb{R}^n$-action). This proves, in particular, that an $\mathrm{sl}_{n+1}$-equivariant symbol is given by a differential map.

**1.5 Remarks**.

(a) The relationship between differential operators and projective geometry has already been studied in the fundamental book [19]. The best known example is the Sturm-Liouville operator $d^2/dx^2 + u(x)$ describing a projective structure on $\mathbf{R}$ (or on $S^1$ if $u(x)$ is periodic).

(b) In the one-dimensional case ($n = 1$), the $\mathrm{sl}_2$-equivariant symbol map and quantization map were obtained (in a more general situation of pseudodifferential operators) in recent work by P.B. Cohen, Yu. I. Manin and D. Zagier [4]. If $n = 1$, our isomorphisms coincide with those of [4]. This article is a revised version of the electronic preprint [14]; we were not aware of the article [4] while the computation of the $\mathrm{sl}_{n+1}$-equivariant symbol was carried out.

(c) In the (algebraic) case of global differential operators on $\mathbb{CP}^n$, existence and uniqueness of the $\mathrm{sl}_{n+1}$-equivariant symbol is a corollary of Borho-Brylinski's results [3]: $\mathcal{D}(\mathbf{CP}^n)$, as a module over $\mathrm{sl}(n+1,\mathbb{C})$, has a decomposition as a sum of irreducible submodules of *multiplicity one*. This implies the uniqueness result. Our explicit formulæ are valid in the holomorphic case and define an isomorphism between $\mathcal{D}(\mathbb{CP}^n)$ and the space of functions on $T^*\mathbb{CP}^n$ which are polynomial on fibers.

We believe that the appearance of projective symmetries in the context of quantization is natural. We do not know any work on this subject except the special one-dimensional case. A particular role of the Lie algebra of projective transformations is related to the fact that $\mathrm{sl}(n+1,\mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n)$ is a *maximal* Lie subalgebra. Any bigger Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$ is infinite-dimensional.

# 2 Modules of differential operators on $\mathbb{R}^n$

## 2.1 Definition of the $\mathrm{Vect}(\mathbb{R}^n)$-module structures

Let us recall the definition of the natural 1-parameter family of $\mathrm{Vect}(\mathbb{R}^n)$-module structures on the space of differential operators (see [6, 13, 9]).

**Definition.** For each $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), consider the space $\mathcal{F}_\lambda$ of *tensor densities* of degree $\lambda$ on $\mathbb{R}^n$, that is, of sections of the line bundle $|\Lambda^n T^*\mathbb{R}^n|^{\otimes\lambda}$. The space $\mathcal{F}_\lambda$ has a natural structure of $\mathrm{Vect}(\mathbb{R}^n)$-module, defined by the Lie derivative.

In coordinates :

$$\phi = \phi(x^1, \dots, x^n)\left|dx^1 \wedge \dots \wedge dx^n\right|^\lambda .$$

The action of $X \in \mathrm{Vect}(\mathbb{R}^n)$ on $\phi \in C^\infty(\mathbb{R}^n)$ is given by

$$L_X^\lambda(\phi) = X^i \partial_i \phi + \lambda \partial_i X^i \phi \,, \tag{2.1}$$

where $\partial_i = \partial/\partial x^i$. Note, that the formula (2.1) does not depend on the choice of local coordinates.

**Remark.** Modules $\mathcal{F}_\lambda$ are not isomorphic to each other for different values of $\lambda$ (cf. [8]). The simplest examples of modules of tensor densities are $\mathcal{F}_0 = C^\infty(\mathbb{R}^n)$ and $\mathcal{F}_1 = \Omega^n(\mathbb{R}^n)$, the module $\mathcal{F}_{1/2}$ is particularly important for geometric quantization (see [2, 11]).

**Definition.** Consider the space $\mathcal{D}_\lambda(\mathbb{R}^n)$ (or $\mathcal{D}_\lambda$ for short) of differential operators on tensor densities, $A : \mathcal{F}_\lambda \to \mathcal{F}_\lambda$. The natural $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_\lambda$ is given by

$$\mathcal{L}_X^\lambda(A) = L_X^\lambda \circ A - A \circ L_X^\lambda. \tag{2.2}$$

Denote $\mathcal{D}_\lambda^k \subset \mathcal{D}_\lambda$ the $\mathrm{Vect}(\mathbb{R}^n)$-module of $k$-th-order differential operators.

In local coordinates any linear differential operator order $k$ is of the form :

$$A = a_k^{i_1 \dots i_k} \partial_{i_1} \cdots \partial_{i_k} + \cdots + a_1^i \partial_i + a_0 \tag{2.3}$$

with coefficients $a_j^{i_1 \dots i_j} = a_j^{i_1 \dots i_j}(x^1, \dots, x^n) \in C^\infty(\mathbb{R}^n)$ (sum over repeated indices is understood).

## 2.2 $\mathrm{Vect}(\mathbb{R}^n)$-modules of symmetric tensor fields on $\mathbb{R}^n$

Consider the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ of functions on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ polynomial on the fibers: $P = \sum_{j=0}^k a_j^{i_1 \dots i_j} \xi_{i_1} \cdots \xi_{i_j}$ . This space has a natural $\mathrm{Vect}(\mathbb{R}^n)$-module structure (defined by the lift of a vector field to the cotangent space) :

$$L_X = X^i \partial_i - \xi_j \partial_i X^j \partial_{\xi_i} \,, \tag{2.4}$$

here and below we denote $\partial_i = \partial/\partial x^i$ and $\partial_{\xi_i} = \partial/\partial \xi_i$.

As a $\mathrm{Vect}(\mathbb{R}^n)$-module, $\mathrm{Pol}(T^*\mathbb{R}^n)$ is isomorphic to the direct sum of the modules of symmetric contravariant tensor fields on $\mathbb{R}$, i.e. $\mathrm{Pol}^k(T^*\mathbb{R}^n) \cong \mathcal{S}^0 \oplus \cdots \oplus \mathcal{S}^k$, where $\mathcal{S}^\ell = \Gamma(S^\ell(T\mathbb{R}^n))$.

## 2.3 Identification of the vector spaces $\mathcal{D}_\lambda$ and $\mathrm{Pol}(T^*\mathbb{R}^n)$

**Definition.** Substituting the monomial $\xi_{i_1} \cdots \xi_{i_j}$ (where $\xi = (\xi_i) \in \mathbb{R}^{n*}$) to the partial derivation $\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_j}}$, allows us to identify $A$ with some element of $\mathrm{Pol}(T^*\mathbb{R}^n)$ :

$$a_k^{i_1 \dots i_k} \partial_{i_1} \cdots \partial_{i_k} \mapsto a_k^{i_1 \dots i_k} \xi_{i_1} \cdots \xi_{i_k} \tag{2.5}$$

We get in this way an *isomorphism of vector spaces* $\mathcal{D} \cong \mathrm{Pol}(T^*\mathbb{R}^n)$.

The $\mathrm{Vect}(\mathbb{R}^n)$-action (2.2) is, of course, different from the standard $\mathrm{Vect}(\mathbb{R}^n)$-action (2.4) on polynomials. We will, therefore, distinguish two $\mathrm{Vect}(\mathbb{R}^n)$-modules :

$$\mathcal{D}_\lambda \equiv (\mathrm{Pol}(T^*\mathbb{R}^n), \mathcal{L}^\lambda), \tag{2.6}$$

$$\mathcal{S} \equiv (\mathrm{Pol}(T^*\mathbb{R}^n), L). \tag{2.7}$$

In particular, a vector field $X$ corresponds to a first-order polynomial: $X = X^i \xi_i$. The operator of Lie derivative is then given by a Hamiltonian vector field

$$L_X = \partial_{\xi_i} X \partial_i - \partial_i X \partial_{\xi_i} \,. \tag{2.8}$$

**Remark.** The identification (2.5) is often called in mathematical physics the *normal ordering*. Another frequently used way to identify the spaces of differential operators on $\mathbb{R}^n$ with the space $\mathrm{Pol}(T^*\mathbb{R}^n)$ is provided by the Weyl symbol calculus.

## 2.4 Comparison of the $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_\lambda$ and $\mathcal{S}$

Let us compare the $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_\lambda$ with the standard $\mathrm{Vect}(\mathbb{R}^n)$-action (2.8) on $\mathrm{Pol}(T^*\mathbb{R}^n)$. We will use the identification (2.5) and write the $\mathrm{Vect}(\mathbb{R}^n)$-action (2.2) in terms of polynomials.

**Lemma 2.1** *The* $\mathrm{Vect}(\mathbb{R}^n)$*-action on* $\mathcal{D}_\lambda$ *has the following form :*

$$\begin{aligned} \mathcal{L}_X^\lambda &= L_X - \tfrac{1}{2}\,\partial_{ij} X \partial_{\xi_i \xi_j} - \lambda\,(\partial_i \circ \mathrm{Div}) X \partial_{\xi_i} \\ &\quad + (\textit{higher order derivatives } \partial_{i_1} \cdots \partial_{i_l} X)\,, \end{aligned} \tag{2.9}$$

*where* $\mathrm{Div}\, X = \partial_i \partial_{\xi_i} X = \partial_i X^i$.

*Proof.* Straightforward computation. ■

# 3 Projective Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$

The standard action of the Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ on $\mathbb{R}^n$ is generated by the vector fields :

$$\partial_i\,, \quad x^i\partial_j\,, \quad x^i\mathcal{E}\,, \tag{3.1}$$

where

$$\mathcal{E} = x^j\partial_j\,. \tag{3.2}$$

It will be called the *projective Lie algebra*. It contains the *affine Lie algebra*, the Lie subalgebra generated by the constant and the linear vector fields.

**Remark**. The group $\mathrm{SL}(n+1,\mathbb{R})$ acts on $\mathbb{RP}^n$ by homographies (linear-fractional tranformations). This action is locally defined on $\mathbb{R}^n \subset \mathbb{RP}^n$. The Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$ tangent to this action coincides with (3.1).

## 3.1 Maximality of $\mathrm{sl}(n+1,\mathbb{R})$ inside $\mathrm{Vect}(\mathbb{R}^n)$

It seems to be quite a known fact that the projective Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ is a maximal subalgebra of the Lie algebra of polynomial vector fields on $\mathbb{R}^n$ (cf. [10] for $n=1,2$.) For the sake of completeness, we will prove here the following version of this result, supposing for simplicity that $n \geq 3$.

**Proposition 3.1** *Given an arbitrary polynomial vector field $X \notin \mathrm{sl}(n+1,\mathbb{R})$, the Lie algebra generated by $\mathrm{sl}(n+1,\mathbb{R})$ and $X$ is the Lie algebra of all polynomial vector fields on $\mathbb{R}^n$.*

*Proof.* We will need the following lemmas.

**Lemma 3.2** *Given a polynomial vector field $X$ such that for every $i=1,\ldots,n$ one has: $[\partial_i, X] = \partial_i X \in \mathrm{sl}(n+1,\mathbb{R})$, each component of the vector field $X$ is a polynomial in $x^1,\ldots,x^n$ of degree at most 2.*

*Proof.* Suppose that $\deg X \geq 3$ and that $\partial_i X = \alpha_i \mathcal{E}$, where $\alpha_i$ are some linear functions and $\mathcal{E}$ is the Euler field (3.2). From $\partial_i\partial_j = \partial_j\partial_i$, one has: $(\partial_i\alpha_j - \partial_i\alpha_j)\mathcal{E} + \alpha_j\partial_i - \alpha_i\partial_j = 0$. If $n \geq 3$, this system has a unique solution $\alpha_1 = \cdots = \alpha_n \equiv 0$. ■

We will also need the following classical fact :

**Lemma 3.3** *The space $\mathcal{S}^1_2$ of symmetric 1-contravariant, 2-covariant tensor field on $\mathbb{R}^n$ is split as an $\mathrm{gl}(n,\mathbb{R})$-module into two irreducible components:*

$$\mathcal{S}^1_2 = \mathcal{A} \oplus \mathcal{B}\,, \qquad \textit{where} \quad \mathcal{A} = \mathbb{R}^{n*}\cdot\mathcal{E} \subset \mathrm{sl}(n+1,\mathbb{R}) \quad \textit{and} \quad \mathcal{B} = \mathrm{Ker}(\mathrm{Div})\,,$$

*and where* $\mathrm{Div} = \partial_i\partial_{\xi_i}$.

Let us now prove the proposition. Considering the commutators $[\partial_i, X]$ and using Lemma 3.2, we can suppose that the vector field $X$ is a homogeneous polynomial of degree 2. Then, $X = X_A + X_B$ where $X_A \in \mathcal{A}$ and $X_B \in \mathcal{B}$. By Lemma 3.3, the Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ and $X_B$ generate the space of all second order vector fields on $\mathbb{R}^n$. Proposition 3.5 follows then from the following lemma whose proof is straightforward.

**Lemma 3.4** *If $n \geq 2$, then the space of all vector fields on $\mathbb{R}^n$ polynomial of degree 2 generates the Lie algebra of all polynomial vector fields on $\mathbb{R}^n$.*

Proposition 3.5 is proven. ■

**Corollary 3.5**
*(i) The projective Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ is a maximal subalgebra of the Lie algebra of polynomial vector fields.*
*(ii) The projective Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ is a maximal subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$ in the class of finite-dimensional Lie subalgebras.*

**Remark**. It worth noticing that the problem of classification of finite-dimensional Lie subalgebras of $\mathrm{Vect}(\mathbb{R}^n)$ goes back to S. Lie and remains open (cf. [10]). The only known examples of maximal semisimple Lie subalgebras of $\mathrm{Vect}(\mathbb{R}^n)$ are the projective Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ and the conformal Lie algebra $\mathrm{o}(p+1,q+1)$ with $p+q=n$. The second case was studied in [7] as a continuation of [14].

### 3.2 Action of $\mathrm{sl}(n+1,\mathbb{R})$ on $\mathcal{D}_\lambda$

The action (2.2) restricted to $\mathrm{sl}(n+1,\mathbb{R})$ takes a particularly nice form.

**Proposition 3.6** *The action of $\mathrm{sl}(n+1,\mathbb{R})$ on $\mathcal{D}_\lambda$ reads :*

$$\mathcal{L}^\lambda_X = L_X, \tag{3.3}$$

$$\mathcal{L}^\lambda_{X_s} = L_{X_s} - \Big(\mathrm{E} + \lambda(n+1)\Big)\partial_{\xi_s} \tag{3.4}$$

*for $X$ from the affine subalgebra of $\mathrm{sl}(n+1,\mathbb{R})$ and $X_s$ a quadratic vector field given by (3.1) and where*

$$\mathrm{E} = \xi_i \partial_{\xi_i}\,. \tag{3.5}$$

*Proof.* Straightforward in view of Lemma 2.1. ■

The formula (3.3) implies that the identification (2.5) is an isomorphism between $\mathrm{Pol}(T^*\mathbb{R}^n)$ and $\mathcal{D}_\lambda$ as modules over the affine Lie algebra.

## 4 Projectively equivariant symbol map

**Definition**.

(a) A linear bijection $\sigma : \mathcal{D}_\lambda \to \mathrm{Pol}(T^*\mathbb{R}^n)$ is called a *symbol map* if for every $A \in \mathcal{D}_\lambda$, the highest order term of $\sigma(A)$ coincides with the *principal symbol*

$$\sigma_A = a_k^{i_1\dots i_k}\xi_{i_1}\cdots\xi_{i_k}. \tag{4.1}$$

(b) We will say that $\sigma$ is *differential* if in addition its restriction to each homogeneous component of $\mathcal{D}_\lambda$ is a differential operator.

**Remark**. The coefficients $a_j^{i_1\dots i_j}$ of the differential operator (2.3) have no geometric meaning, except for $j = k$. In other words, there is no natural symbol map, i.e. there is no linear bijection from $\mathcal{D}_\lambda$ into the space of symmetric contravariant tensor fields on $\mathbb{R}^n$ equivariant with respect to the $\mathrm{Vect}(\mathbb{R}^n)$-action.

We will see below that for $\lambda = 1/2$, there is also a well-defined symbol of order $k-1$ on $\mathcal{D}^k_{1/2}$.

### 4.1 The divergence operator as an affine invariant

Let us introduce a differential operator on the space of polynomials $\mathcal{S} \cong \mathrm{Pol}(T^*\mathbb{R}^n)$ (and on $\mathcal{D}_\lambda$ using the identification (2.5)) :

$$\mathrm{Div} = \partial_i \partial_{\xi_i} \,. \tag{4.2}$$

On a homogeneous component of order $k$, $\mathcal{S}^k \subset \mathcal{S}$ one has :

$$\mathrm{Div}\Big|_{\mathcal{S}^k} = k\,\mathrm{div}\,, \tag{4.3}$$

where

$$\Big(\mathrm{div}\,(a_k)\Big)^{i_1\dots i_{k-1}} = \partial_j(a_k^{i_1\dots i_{k-1}j})\,. \tag{4.4}$$

**Remarks**. (a) The operator (4.2) *commutes with the action of the affine Lie algebra.* Moreover, it follows from the classical Weyl-Brauer theorem that any linear operator on $\mathcal{S}$ commuting with the affine Lie algebra is a polynomial expression in Div and E defined by (3.5).

(b) Fix the standard volume form $\Omega = dx^1 \wedge \dots \wedge dx^n$ on $\mathbb{R}^n$, then, for any vector field $X$, the standard divergence $\mathrm{div}_\Omega(X)$ coincides with $\mathrm{Div}\,X$ (after the identification (2.5)).

### 4.2 Statement of the main result

The main result of this paper is the existence and uniqueness of an $\mathrm{sl}_{n+1}$-equivariant symbol map on $\mathbb{R}^n$. We will prove that such a map is necessarily differential.

**Theorem 4.1** *For every $\lambda$, there exists a unique $\mathrm{sl}_{n+1}$-equivariant symbol map $\sigma_\lambda$. It is differential and given on a homogeneous component $\mathcal{S}^k$ by*

$$\sigma_\lambda\Big|_{\mathcal{S}^k} = \sum_{\ell \le k} c^k_\ell\,\mathrm{div}^{k-\ell} \tag{4.5}$$

*where the numbers $c^k_\ell$ are as follows:*

$$c^k_\ell = (-1)^{k-\ell}\frac{\binom{k}{\ell}\binom{(n+1)\lambda+k-1}{k-\ell}}{\binom{k+\ell+n}{k-\ell}} \tag{4.6}$$

Recall that the binomial coefficient $\binom{\alpha}{m}$ for $\alpha \notin \mathbb{N}$ is $\frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}$.

**Remarks**.

(a) The condition of $\mathrm{sl}_{n+1}$-equivariance is already sufficient to determine $\sigma_\lambda|_{\mathcal{S}^k}$ up to a constant. The supplementary condition, that the higher order term of the polynomial $\sigma_\lambda(A)$ coincides with the principal symbol $\sigma_A$, fixes the normalization.

(b) In the one-dimensional case ($n = 1$) the formula (4.5,4.6) is also valid for the space of pseudodifferential operators. In this case, it coincides with (4.11) of [4].

### 4.3 Example: second order operators and quadratic Hamiltonians

Let us apply the general formulæ (4.5,4.6) to second order differential operators.

The $\mathrm{sl}_{n+1}$-equivariant symbol map associates to a second-order differential operator $A = a_2^{ij}\partial_i\partial_j + a_1^i\partial_i + a_0$ the polynomial $\sigma_\lambda(A) = \bar{a}_2^{ij}\xi_i\xi_j + \bar{a}_1^i\xi_i + \bar{a}_0$, with the coefficients :

$$\begin{aligned}
\bar{a}_2^{ij} &= a_2^{ij} \\
\bar{a}_1^i &= a_1^i - 2\frac{(n+1)\lambda+1}{n+3}\partial_j(a_2^{ij}) \\
\bar{a}_0 &= a_0 - \lambda\partial_i(a_1^i) + \lambda\frac{(n+1)\lambda+1}{n+2}\partial_i\partial_j(a_2^{ij})\,.
\end{aligned}$$

### 4.4 Proof of Theorem 4.1

Let us first prove that symbol map $\sigma_\lambda$ defined by the formulæ (4.5) and (4.6) is $\mathrm{sl}_{n+1}$-equivariant.

A symbol map (4.5) with arbitrary constants $c_\ell^k$, is obviously equivariant with respect to the affine algebra. It can be rewritten in the form :

$$\sigma_\lambda\Big|_{\mathcal{S}^k} = \sum_{\ell=O}^{k} C_\ell^k\,\mathrm{Div}^{k-\ell}\,, \tag{4.7}$$

where $c_\ell^k = k(k-1)\cdots(k-\ell+1)C_\ell^k$. Now, to determine the constants $C_\ell^k$, one needs the following commutation relations.

**Lemma 4.2** *For $X_s = x^s\mathcal{E} \in \mathrm{sl}(n+1,\mathbb{R})$, one has :*

$$[L_{X_s}, \mathrm{Div}] = \Big(2\mathrm{E} + (n+1)\Big)\circ\partial_{\xi_s}\,, \tag{4.8}$$

*Proof.* Straightforward. ■

Let us calculate a recurrent relation for the coefficients $C_\ell^k$ in (4.7). The condition of $\mathrm{sl}(n+1,\mathbb{R}^n)$-equivariance reads :

$$L_X\circ\sigma_\lambda = \sigma_\lambda\circ\mathcal{L}_X^\lambda\,,\quad \text{for every}\quad X\in\mathrm{sl}(n+1,\mathbb{R}^n)\,.$$

For $X = X_s$, from (3.4) one, therefore, gets :

$$[L_{X_s},\sigma_\lambda] = -\sigma_\lambda\circ\Big(\mathrm{E}+\lambda(n+1)\Big)\circ\partial_{\xi_s}. \tag{4.9}$$

We are looking for the symbol map $\sigma_\lambda$ in the form (4.7). From the equation (4.7,4.9) one immediately obtains :

$$C_\ell^k \left[L_X, \mathrm{Div}^{k-\ell}\right]\Big|_{\mathcal{S}^k} = -C_\ell^{k-1}\,\mathrm{Div}^{k-\ell-1} \circ \Big(\mathrm{E} + \lambda(n+1)\Big) \circ \partial_{\xi_s}.$$

From the commutation relation (4.8) and using that $\mathrm{E} = k\,\mathrm{Id}$ on $\mathcal{S}^k$, one easily gets :

$$[L_{X_s}, \mathrm{Div}^m]\Big|_{\mathcal{S}^k} = m(2k-m+n)\,\mathrm{Div}^{m-1} \circ \partial_{\xi_s}\,,$$

and, finally, one obtains the recurrent relation :

$$C_\ell^k = -\frac{k-1+\lambda(n+1)}{(k-\ell)(k+\ell+n)} C_\ell^{k-1}\,. \tag{4.10}$$

This together with $C_k^k = 1$ is equivalent to (4.6).

This proves that the symbol map $\sigma_\lambda$ is, indeed, $\mathrm{sl}_{n+1}$-equivariant.

The uniqueness of the $\mathrm{sl}_{n+1}$-equivariant map $\sigma_\lambda$ follows from :

**Proposition 4.3** *The only* $\mathrm{sl}_{n+1}$*-equivariant linear maps on the space of symmetric contravariant tensor fields are multiplications by constants, namely :*

$$\mathrm{Hom}_{\mathrm{sl}_{n+1}}(\mathcal{S}^k, \mathcal{S}^\ell) = \begin{cases} \mathbb{R}\,, & k=\ell \\ 0\,, & k \neq \ell \end{cases} \tag{4.11}$$

*Proof.* It is easy to compute the Casimir operator of $\mathrm{sl}(n+1,\mathbb{R})$ :

$$C\Big|_{\mathcal{S}^k} = 2k(k+n)\,\mathrm{Id}\,. \tag{4.12}$$

Since $C$ commutes with the $\mathrm{sl}(n+1,\mathbb{R})$-action, this implies the result (4.11) in the case $k \neq \ell$.

In the case $k=\ell$, the result easily follows from Theorem 5.1 below. Moreover, Lemma 5.2 implies that an $\mathrm{sl}_{n+1}$-equivariant linear operator on $\mathcal{S}^k$ is necessarily a differential operator of order zero. Then, it follows from the classical Weyl-Brauer theorem that it is proportional to the identity. ∎

Theorem 4.1 is proven. ∎

### 4.5 Projectively equivariant quantization

The inverse to the symbol map, $\sigma_\lambda^{-1}$, is the unique (up to normalization) $\mathrm{sl}_{n+1}$-equivariant quantization map.

**Proposition 4.4** *The map* $\sigma_\lambda^{-1}$ *is defined by :*

$$\sigma_\lambda^{-1}\Big|_{\mathcal{S}^k} = \sum_{\ell \le k} \bar{c}_\ell^k\,\mathrm{div}^{k-\ell} \tag{4.13}$$

*where*

$$\bar{c}_\ell^k = \frac{\binom{k}{\ell}\binom{(n+1)\lambda+k-1}{k-\ell}}{\binom{2k+n-1}{k-\ell}}\,. \tag{4.14}$$

*Proof.* Applying the same method as in the proof of Theorem 4.1, one can show that there exists a unique differential operator $Q_\lambda : \mathrm{Pol}(T^*\mathbb{R}^n) \to \mathcal{D}_\lambda$ which is $\mathrm{sl}(n+1,\mathbb{R})$-equivariant and such that the principal symbol of $Q_\lambda(P)$ coincides with the leading term of $P$, namely, $Q_\lambda = \sigma_\lambda^{-1}$. On the space of homogeneous polynomials this map is given by

$$Q_\lambda \Big|_{\mathcal{S}^k} = \sum_{\ell \leq k} \bar{C}_\ell^k \, \mathrm{Div}^{k-\ell} . \tag{4.15}$$

The numbers $\bar{C}_\ell^k$ are defined by the relation:

$$\begin{cases} \bar{C}_\ell^k & = & \dfrac{\ell + (n+1)\lambda}{(k-\ell)(k+\ell+n)} \, \bar{C}_{\ell+1}^k \\ \bar{C}_k^k & = & 1 \end{cases}$$

The result follows. ■

## 4.6 Example: quantization of the geodesic flow

Consider a nondegenerate quadratic form $H = g^{ij}\xi_i\xi_j$ and apply the $\mathrm{sl}_{n+1}$-equivariant quantization map in the special case of $\lambda = 1/2$. It is easy to check that the result of quantization (4.13) is a *Laplace-Beltrami operator*. Namely :

$$Q_{1/2}(H) = \Delta_g + \frac{(n+1)}{4(n+2)} \partial_{ij} g^{ij} , \tag{4.16}$$

where $\Delta_g = g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k \partial_k)$ is the standard Laplace operator corresponding to the metric $g = H^{-1} = g_{ij}dx^i dx^j$.

Recall that in a neighborhood of each point $u$, there exist so-called *normal coordinates*, that is coordinates characterized by: $\Gamma_{ij}^k(u) = 0$ and $\partial_l(\Gamma_{ij}^k) = \frac{1}{3}(R_{li,j}^k + R_{lj,i}^k)$. In these coordinates, the potential of (4.16) is proportional to the scalar curvature: $\partial_{ij} g^{ij} = (1/3)R$.

One obtains the following result.

**Proposition 4.5** *In the case when the normal coordinates of $g$ are compatible with the projective structure, the quantum hamiltonian is given by*

$$Q_{1/2}(H) = \Delta_g + \frac{(n+1)}{12(n+2)} R . \tag{4.17}$$

**Remark**. The problem of quantization of the geodesic flow on a (pseudo)-Riemannian manifold have already been considered by many authors (see [5] end references therein). Various methods leads to formulæ of the type (4.17) but with different values of the multiple in front of the scalar curvature. The formulæ (4.16,4.17) is a new version of the quantization of the geodesic flow.

## 5 Equivariance and locality

Consider the Lie algebra $\mathbb{R} \ltimes \mathbb{R}^n$ generated by the vector fields

$$\mathcal{E} = x^i \frac{\partial}{\partial x^i}, \quad \frac{\partial}{\partial x^1}, \quad \ldots, \quad \frac{\partial}{\partial x^1}, \tag{5.1}$$

(which is, of course, a subalgebra of $\mathrm{sl}(n+1, \mathbb{R})$).

The following result seems to be quite unexpected.

**Theorem 5.1** *If $p \geq q$, then every linear map $T : \mathcal{S}^p \to \mathcal{S}^q$ equivariant with respect to the Lie algebra generated by the vector fields (5.1) is local.*

*Proof.* Assume that $P \in \mathcal{S}^p$ vanishes in a neighborhood of some point $u \in \mathbb{R}^n$. We will prove that $T(P)_u = 0$. Using invariance with respect to translations $\mathbb{R}^n \subset \mathrm{sl}(n+1, \mathbb{R})$, we may assume that $u = 0$.

**Lemma 5.2** *For every $P \in \mathcal{S}^p$ such that the $(s-1)$-jet of $P$ at the origin vanishes, there exists $Q \in \mathcal{S}^p$ such that*

$$P = x^{i_1} \cdots x^{i_s} \partial_{i_1} \cdots \partial_{i_s}(Q)$$

*Proof of the lemma.* It follows directly from the Taylor integral formula. ■

In terms of $\mathcal{E} = x^i \partial_i$, this reads:

$$P = (\mathcal{E} - (s-1)\,\mathrm{Id})(\mathcal{E} - (s-2)\,\mathrm{Id}) \cdots (\mathcal{E} - \mathrm{Id})\mathcal{E}(Q).$$

Taking account of the fact that the Lie derivative $L_{\mathcal{E}}$ on $\mathcal{S}^p$ is given by $L_{\mathcal{E}} = \mathcal{E} - p\,\mathrm{Id}$, we thus obtain:

$$P = (L_{\mathcal{E}} + (p-s+1)\,\mathrm{Id})(L_{\mathcal{E}} + (p-s+2)\,\mathrm{Id}) \cdots (L_{\mathcal{E}} + (p-1)\,\mathrm{Id})(L_{\mathcal{E}} + p\,\mathrm{Id})(Q).$$

Therefore, by $\mathrm{sl}_{n+1}$-equivariance,

$$\begin{aligned} T(P) &= (L_{\mathcal{E}} + (p-s+1)\,\mathrm{Id}) \cdots (L_{\mathcal{E}} + p\,\mathrm{Id})T(Q) \\ &= (\mathcal{E} + (p-q-s+1)\,\mathrm{Id}) \cdots (\mathcal{E} + (p-q)\,\mathrm{Id})T(Q). \end{aligned}$$

With $s = p - q + 1$, we obtain $T(P)_0 = 0$. Hence the result. ■

**Remarks**.

(a) By the well-known Peetre theorem [17], the map $T$ from Theorem 5.1 is (locally) a differential operator. This theorem therefore generalizes the statement of Theorem 4.1 that every $\mathrm{sl}_{n+1}$-equivariant symbol map is differential.

(b) The Lie algebra generated by the vector fields (5.1) is also a subalgebra of the conformal Lie algebra $\mathrm{o}(p+1, q+1)$, and, therefore, can be implied in this case.

# 6 Action of $\mathrm{Vect}(\mathbb{R}^n)$ on $\mathcal{D}_\lambda$ in $\mathrm{sl}_{n+1}$-equivariant form

We now turn to study the space of differential operators on $\mathbb{R}^n$ as a module over the Lie algebra $\mathrm{Vect}(\mathbb{R}^n)$ of all vector fields on $\mathbb{R}^n$. Using the $\mathrm{sl}_{n+1}$-equivariant symbol $\sigma_\lambda$ one can obtain a canonical (projectively invariant) form of the $\mathrm{Vect}(\mathbb{R}^n)$-action on $\mathcal{D}_\lambda$ :

$$\begin{array}{ccc} \mathcal{D}_\lambda & \xrightarrow{\mathcal{L}_X^\lambda} & \mathcal{D}_\lambda \\ {\scriptstyle\sigma_\lambda}\Big\downarrow & & \Big\downarrow{\scriptstyle\sigma_\lambda} \\ \mathrm{Pol}(T^*\mathbb{R}^n) & \xrightarrow{\sigma_\lambda\circ\mathcal{L}_X^\lambda\circ\sigma_\lambda^{-1}} & \mathrm{Pol}(T^*\mathbb{R}^n) \end{array} \tag{6.1}$$

This action is clearly of the form

$$\sigma_\lambda \circ \mathcal{L}_X^\lambda \circ \sigma_\lambda^{-1} = L_X + \sum_{\ell\geq 1} \gamma_\ell^\lambda(X)\,, \tag{6.2}$$

where $\gamma_\ell^\lambda$ are linear maps associating to $X$ some operators on $\mathrm{Pol}(T^*\mathbb{R}^n)$, namely, for every $k$,

$$\gamma_\ell^\lambda : \mathrm{Vect}(\mathbb{R}^n) \to \mathrm{Hom}(\mathcal{S}^k, \mathcal{S}^{k-\ell})\,. \tag{6.3}$$

The operators $\gamma_\ell^\lambda$ play important role in the study of these modules.

## 6.1 Elementary properties of the maps $\gamma_\ell^\lambda$

The following lemma is an immediate corollary of the $\mathrm{sl}_{n+1}$-equivariance of $\sigma_\lambda$ :

**Lemma 6.1** *The linear maps (6.3) satisfy the following properties :*
*(a)* $\mathrm{sl}_{n+1}$*-equivariance:*

$$[L_X, \gamma_\ell^\lambda(Y)] = \gamma_\ell^\lambda([X,Y]), \quad X \in \mathrm{sl}_{n+1}\,, \tag{6.4}$$

*(b) vanishing on* $\mathrm{sl}_{n+1}$*:*

$$\gamma_\ell^\lambda(X) = 0, \quad X \in \mathrm{sl}_{n+1}. \tag{6.5}$$

The next statement follows directly from the fact that (6.2) is an action of $\mathrm{Vect}(\mathbb{R}^n)$ :

**Lemma 6.2** *For every $\lambda$, the map $\gamma_1^\lambda$ is a 1-cocycle on* $\mathrm{Vect}(\mathbb{R}^n)$*, i.e., it satisfies the relation :*

$$[L_X, \gamma_1^\lambda(Y)] - [L_Y, \gamma_1^\lambda(X)] = \gamma_1^\lambda([X,Y])\,, \tag{6.6}$$

*for every $X, Y \in$* $\mathrm{Vect}(\mathbb{R}^n)$*.*

Moreover, in the case when $\gamma_1^\lambda$ vanishes, then the cocycle property holds true for $\gamma_2^\lambda$.

**Remark**. These simple facts explain how the modules of differential operators are related to so-called $\mathrm{sl}(n+1,\mathbb{R})$-relative cohomology of $\mathrm{Vect}(\mathbb{R}^n)$ (that is, $\mathrm{Vect}(\mathbb{R}^n)$-cohomology vanishing on $\mathrm{sl}(n+1,\mathbb{R})$). This subject will be treated in a subsequent article.

We will need explicit formulæ for $\gamma_1^\lambda$ and $\gamma_2^\lambda$.

## 6.2 Computing $\gamma_1^\lambda$

**Proposition 6.3** *One has :*

$$\gamma_1^\lambda\Big|_{\mathcal{S}^k} = \frac{(n+1)}{2(2k+n-1)}\,(2\lambda-1)\,\ell_k\,, \tag{6.7}$$

*where* $\ell_k : \mathrm{Vect}(\mathbb{R}^n) \to \mathrm{Hom}(\mathcal{S}^k, \mathcal{S}^{k-1})$ *is the following operator :*

$$\ell_k(X) = \partial_{ij}X\,\partial_{\xi_i\xi_j} - \frac{2(k-1)}{n+1}\,(\partial_i \circ \mathrm{Div})X\,\partial_{\xi_i}\,. \tag{6.8}$$

*Proof.* Applying explicit formulæ (4.15), (2.9) and (4.7) for each factor in $\sigma_\lambda \circ \mathcal{L}_X^\lambda \circ \sigma_\lambda^{-1}$, one immediately obtains :

$$\begin{array}{rcl}\gamma_1^\lambda(X)\Big|_{\mathcal{S}^k} & = & \bar{C}_{k-1}^k\,L_X \circ \mathrm{Div} + C_{k-1}^k\,\mathrm{Div} \circ L_X \\ & & -\frac{1}{2}\,\partial_{ij}X\,\partial_{\xi_i\xi_j} - \lambda\,(\partial_i \circ \mathrm{Div})X\,\partial_{\xi_i} \\ & = & -(C_{k-1}^k - \frac{1}{2})\,\partial_{ij}X\,\partial_{\xi_i\xi_j} - (C_{k-1}^k - \lambda)\,(\partial_i \circ \mathrm{Div})X\,\partial_{\xi_i}\,.\end{array}$$

The formula (6.7,6.8) follows. ■

**Important Remark**. The operator (6.8) is, in fact, an operator of contraction with the symmetric $(2,1)$-tensor field: $\ell_k(X)(P) = \langle \bar{\ell}(X), P\rangle$, where :

$$\bar{\ell}(X)_{ij}^h = \partial_{ij}X^h - \frac{1}{n+1}\left(\delta_i^h\partial_j + \delta_j^h\partial_i\right)\partial_l X^l \tag{6.9}$$

This expression is a 1-cocycle on $\mathrm{Vect}(\mathbb{R}^n)$ vanishing on the subalgebra $\mathrm{sl}(n+1,\mathbb{R})$ (cf. [16, 15] and references therein).

## 6.3 Computing $\gamma_2^\lambda$

**Proposition 6.4** *The map* $\gamma_2^\lambda$ *in (6.2) is given by*

$$\gamma_2^\lambda\Big|_{\mathcal{S}^k} = \frac{1}{2(2k+n-2)(2k+n-3)}\,s_k^\lambda,$$

*where the operator* $s_k^\lambda : \mathrm{Vect}(\mathbb{R}^n) \to \mathrm{Hom}(\mathcal{S}^k, \mathcal{S}^{k-2})$ *is :*

$$\begin{array}{rcl}s_k^\lambda(X) & = & \alpha_1\,\partial_{hij}X\,\partial_{\xi_h\xi_i\xi_j} + \alpha_2\,\partial_{ij}X\,\mathrm{Div}\circ\partial_{\xi_i\xi_j} \\ & & +\beta_1\,(\partial_{ij}\circ\mathrm{Div})X\,\partial_{\xi_i\xi_j} + \beta_2\,(\partial_i\circ\mathrm{Div})X\,\mathrm{Div}\circ\partial_{\xi_i} \\ & & +\delta\,\partial_{ij}\partial_{\xi_h}X\,\partial_h\partial_{\xi_i\xi_j}\end{array} \tag{6.10}$$

*with the numerical coefficients :*

$$
\begin{aligned}
\alpha_1 &= -\Big((n+1)^2\lambda(\lambda-1) + \frac{1}{3}(k^2+kn+n^2-k+n)\Big) \\
\alpha_2 &= -\frac{2(n+1)^2\lambda(\lambda-1)+2k^2+2kn-4k+n^2-n+2}{2k+n-1} \\
\beta_1 &= \frac{(4k+n-5)(n+1)\lambda(\lambda-1)-(k-2)(k-1)}{2k+n-1} \\
\beta_2 &= (4k-6)(n+1)\lambda(\lambda-1)+(k-2)n \\
\delta_2 &= -(n+1)^2\lambda(\lambda-1)+(k-2)(k+n-1)
\end{aligned}
\tag{6.11}
$$

*Proof.* A quite complicated straightforward computation similar to that of the proof of Proposition 6.3. ■

## 6.4 Example: case $k = 2$

For $k = 2$, one verifies that the operator (6.10) is as follows:

$$s_2^\lambda(X) = -(n+1)(n+2)\,\lambda(\lambda-1)\,\bar{s}\,,$$

where the operator $\bar{s}$ can be written in terms of the operator (6.8) :

$$\bar{s}(X) = \partial_{\xi_i}\left(\ell_2(X)\right)\,\partial_i - \frac{2}{n-1}\,\mathrm{Div}\left(\ell_2(X)\right)\,. \tag{6.12}$$

**Remark**. In the case $\lambda = 1/2$, the map $\gamma_1$ is identically zero and the first non-zero term, namely $\gamma_2$, in (6.2) is a 1-cocycle on $\mathrm{Vect}(\mathbb{R}^n)$ (cf. Section 6.1).

## 6.5 Conjugation and the $\mathrm{sl}_{n+1}$-equivariant symbol map

Recall from [6] and [13] that the conjugation is an isomorphism of $\mathrm{Vect}(\mathbb{R}^n)$-modules

$$* : \mathcal{D}_\lambda \cong \mathcal{D}_{1-\lambda}\,,$$

(which also exists for an arbitrary manifold $M$, see below). It is characterized by $*(\mathrm{Id}) = \mathrm{Id}$ and by:

$$*\,(L_X^\lambda \circ A) = -\,*\,(A) \circ L_X^{1-\lambda} \tag{6.13}$$

for all $X \in \mathrm{Vect}(\mathbb{R}^n)$ and all $A \in \mathcal{D}_\lambda$.

A nice property of the $\mathrm{sl}_{n+1}$-equivariant symbol map is the following

**Lemma 6.5** *For each homogeneous polynomial $P \in \mathcal{S}^k$ one has : $\sigma_\lambda \circ * \circ \sigma_\lambda^{-1}(P) = (-1)^k P$.*

*Proof.* By Proposition 4.3, the map $\sigma_\lambda \circ * \circ \sigma_\lambda^{-1}$ is an operator of multiplication by a constant. On the other hand, (6.13) shows that the principal symbol $\sigma_{*(A)} = (-1)^k \sigma_A$. ■

**Corollary 6.6** *All the bilinear maps $\gamma_{2p+1}^{\frac{1}{2}}$ vanish.*

# 7 Space of differential operators on a manifold as a module over the Lie algebra of vector fields

Let $M$ be a smooth manifold. Consider the space $\mathcal{F}_\lambda$ of tensor densities of degree $\lambda$ on $M$ and $\mathcal{D}_\lambda(M)$ be the space of scalar linear differential operators $A : \mathcal{F}_\lambda \to \mathcal{F}_\lambda$. The space $\mathcal{D}_\lambda(M)$ has a natural $\mathrm{Vect}(M)$-module structure. The 1-parameter family of $\mathrm{Vect}(M)$-modules $\mathcal{D}_\lambda$ has been recently studied in [6],[13],[9].

In this section we will study the quotient modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$. Our purpose is to solve the problem of isomorphism between these modules for different values of $\lambda$ and to compare these modules with the modules of symmetric contravariant tensor fields $\mathrm{Pol}^k(T^*M)/\mathrm{Pol}^\ell(T^*M) = \mathcal{S}^k(M) \oplus \ldots \oplus \mathcal{S}^{\ell+1}(M)$.

## 7.1 Locality of equivariant maps

Equivariance with respect to $\mathrm{Vect}(M)$ is of course a much stronger condition that equivariance with respect to $\mathrm{sl}(n+1,\mathbb{R})$ or $\mathbb{R} \ltimes \mathbb{R}^n$ (cf. Section 6.4). It has already been shown in [13] that a $\mathrm{Vect}(M)$-equivariant linear map from $\mathcal{D}^k_\lambda(M)$ into $\mathcal{D}^k_\mu(M)$ is local. In the same spirit we have:

**Proposition 7.1** *Every* $\mathrm{Vect}(M)$*-equivariant linear map $T$ from $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ into $\mathcal{D}^k_\mu(M)/\mathcal{D}^\ell_\mu(M)$ is local ($k > \ell$).*

**Proof.** Suppose that $A \in \mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ vanishes on an open subset $U$ of $M$. Let us show that $T(A)$ vanishes on $U$ as well. Assume in the contrary that $T(A)|_u \neq 0$ for some $u \in U$. One can choose $u$ such that the principal symbol $\sigma_{T(A)}|_u \neq 0$. Then, there exists $X \in \mathrm{Vect}(M)$ with compact support in $U$ such that $(L_X(T(A)))|_u \neq 0$. Hence the contradiction since $L_X(A) = 0$ and $L_X(T(A)) = T(L_X(A))$. To see that such $X$ exists, it suffice to choose $X$ such that $(L_X(S))|_u \neq 0$ where $S \in \mathcal{S}^p(M)$, $\ell < p \leq k$, is the principal symbol of $T(A)$ and $p$ is its order around $u$. This is always possible for $p > 0$. Hence the proposition. ■

The above proof is easily adapted to get the following result that we need for later purpose.

**Proposition 7.2** *Every* $\mathrm{Vect}(M)$*-equivariant linear map $T : \mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M) \to \mathcal{S}^p(M)$ is local ($k > \ell, p > 0$).*

## 7.2 Classification of the modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^{k-2}_\lambda(M)$

The following result gives a classification of the quotient modules for the case $\ell = k - 2$.

**Theorem 7.3** *Assume that $k \geq 2$ and that* $\dim M \geq 2$.
*(i)All the* $\mathrm{Vect}(M)$*-modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^{k-2}_\lambda(M)$, with $\lambda \neq 1/2$, are isomorphic. They are not isomorphic to the direct sum of modules of tensor fields $\mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M)$.*
*(ii) The module of differential operators on half-densities is exceptional since*

$$\mathcal{D}^k_{\frac{1}{2}}(M)/\mathcal{D}^{k-2}_{\frac{1}{2}}(M) \cong \mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M).$$

*Proof.* (i) Having local coordinates in a open subset $U \subset M$ and using the $\mathrm{sl}_{n+1}$-equivariant symbol map, we identify $\mathcal{D}^k_\lambda(M)/\mathcal{D}^{k-2}_\lambda(M)$ over $U$ with $\mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M)$ endowed with the $\mathrm{Vect}(M)$-action

$$\mathcal{L}^\lambda_X\Big|_{\mathcal{S}^k\oplus\mathcal{S}^{k-1}} = L_X + \gamma_1^\lambda(X) \tag{7.1}$$

In view of (6.7), it is clear that if $\lambda, \mu \neq 1/2$, the map

$$(P_k, P_{k-1}) \mapsto \left(P_k,\ \frac{2\mu - 1}{2\lambda - 1}\, P_{k-1}\right) \tag{7.2}$$

is an isomorphism between the $\mathrm{Vect}(\mathbb{R}^n)$-modules :

$$(\mathcal{S}^k \oplus \mathcal{S}^{k-1},\ \sigma_\lambda \circ \mathcal{L}^\lambda_X \circ \sigma_\lambda^{-1}) \longrightarrow (\mathcal{S}^k \oplus \mathcal{S}^{k-1},\ \sigma_\mu \circ \mathcal{L}^\mu_X \circ \sigma_\mu^{-1})\,.$$

It, therefore, defines an isomorphism between the restrictions of the $\mathrm{Vect}(M)$-modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^{k-2}_\lambda(M)$ and $\mathcal{D}^k_\mu(M)/\mathcal{D}^{k-2}_\mu(M)$ to the domain $U$ of coordinates. This isomorphism does not depend on the choice of coordinates because it commutes with the $\mathrm{Vect}(M)$-action: the formula (7.2) does not change under the coordinate transformations. Hence, the local isomorphisms defined on each $U$ glue together to define a global isomorphism.

Let now $\mathcal{D}^k_\lambda(M)/\mathcal{D}^{k-2}_\lambda(M) \to \mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M)$ be a $\mathrm{Vect}(M)$-equivariant map. It is local (Proposition 7.2). Hence, by the Peetre theorem [17], it is locally a differential operator. Expressed like above in terms of the $\mathrm{sl}_{n+1}$-equivariant symbol, it has a diagonal form (Proposition 4.3): $(\bar{a}_k, \bar{a}_{k-1}) \mapsto (\alpha_k \bar{a}_k, \alpha_{k-1}\bar{a}_{k-1})$ for some $\alpha_k, \alpha_{k-1} \in \mathbf{R}$. The fact that $T$ intertwines the action (7.1) and the Lie derivative of tensors on $\mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M)$ implies $\alpha_{k-1}\gamma_1^\lambda(X, \bar{a}_k) = 0$ for all $\bar{a}_k \in \mathcal{S}^k(M)$. Since $\lambda \neq 1/2$, it follows that $\alpha_{k-1} = 0$. Therefore, $T$ is not injective.

Hence Part (i) of Theorem 7.3.

### 7.3 Exceptional case $\lambda = 1/2$

If $\lambda = 1/2$, the term $\gamma_1^\lambda(X, \bar{a}_k)$ vanishes. The $\mathrm{Vect}(M)$-action (6.2) in this case is just the standard action on $\mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M)$.

Hence Theorem 7.3. ■

It follows from this theorem that for $\lambda \neq 1/2$, there is no intrinsically defined subsymbol of a differential operator. However, in the exceptional case of differential operators on 1/2-densities, the two first terms of the $\mathrm{sl}_{n+1}$-equivariant symbol have geometric meaning. One obtains the following remark.

**Corollary 7.4** *The* $\mathrm{sl}_{n+1}$*-equivariant symbol defines a* $\mathrm{Vect}(M)$*-equivariant map*

$$(\sigma_k^{1/2}, \sigma_{k-1}^{1/2}) : \mathcal{D}^k_{1/2}(M) \to \mathcal{S}^k(M) \oplus \mathcal{S}^{k-1}(M).$$

*Proof.* The $\mathrm{Vect}(M)$-module $\mathcal{D}^k_{1/2}(M)$ has a symmetry: the conjugation of operators $A \to A^*$ (cf. [6, 13]). Every $A \in \mathcal{D}^k_{1/2}(M)$ has a decomposition $A = A_0 + A_1$, where $A_0^* = (-1)^k A_0$ and $A_1^* = (-1)^{k-1} A_1$. One easily sees that $\sigma_{1/2}(A)_k$ is the principal symbol of $A_0$ and $\sigma_{1/2}(A)_{k-1}$ that of $A_1$. ■

## 7.4 Modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ in multi-dimensional case

We now turn to the $\mathrm{Vect}(M)$-modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ when $k-\ell \geq 3$ and $\dim M \geq 2$. The following result shows that there is no nontrivial isomorphism between the $\mathrm{Vect}(M)$-modules in this case.

**Theorem 7.5** *Assume that $k-\ell \geq 3$ and $\dim M \geq 2$.*
*(i) If $\lambda \neq \mu$, the modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ and $\mathcal{D}^k_\mu(M)/\mathcal{D}^\ell_\mu(M)$ are isomorphic if and only if $\lambda+\mu=1$.*
*(ii) There is no isomorphism between the modules $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ and the module of tensor fields $\mathcal{S}^k(M)\oplus\ldots\oplus\mathcal{S}^{\ell+1}(M)$.*

*Proof.* The isomorphism in Part (i) is given by the standard conjugation of differential operators. This map defines a general isomorphism $*:\mathcal{D}_\lambda(M)\to\mathcal{D}_{1-\lambda}(M)$ (cf. [6, 13]).

The proof that there is no other isomorphism is very similar to that of Theorem 7.3. In local coordinates and in terms of $\mathrm{sl}_{n+1}$-equivariant symbol, such an isomorphism between $\mathcal{D}^k_\lambda(M)/\mathcal{D}^\ell_\lambda(M)$ and $\mathcal{D}^k_\mu(M)/\mathcal{D}^\ell_\mu(M)$ (or $S^k(M)\oplus\cdots\oplus S^{\ell+1}(M)$) is of the form:

$$(P_k,\ P_{k-1},\ P_{k-2},\ldots)\to(\alpha_k\,P_k,\ \alpha_{k-1}P_{k-1},\ \alpha_{k-2}\,P_{k-2},\ldots) \tag{7.3}$$

for some $\alpha_k,\alpha_{k-1},\alpha_{k-2},\ldots\in\mathbf{R}$.

In the both cases, (i) and (ii), the $\mathrm{Vect}(M)$-equivariance condition now involves not only $\gamma^\lambda_1$ but also $\gamma^\lambda_2$. Tedious computations allow then to show in the first case that (7.3) is an isomorphism if and only if $(\lambda-\mu)(\lambda+\mu-1)=0$. In the second case, equivariance immediately implies that $\alpha_{k-1}=\alpha_{k-2}=0$. ■

Part (ii) of Theorem 7.5 confirms the fact (well-known "in practice") that a "complete" symbol of a differential operator can not be defined in an intrinsic way (even in the case of differential operators on 1/2-densities).

## 7.5 Modules of second order differential operators

Consider the modules of second order differential operators $\mathcal{D}^2_\lambda(M)$.

These modules have been classified in [6] where it has been shown that there are exactly three isomorphism classes of $\mathrm{Vect}(M)$-modules among them, namely

$$\{\mathcal{D}^2_\lambda(M),\ \lambda\neq 0,1/2,1\},\quad \{\mathcal{D}^2_0(M),\mathcal{D}^2_1(M)\}\quad\text{and}\quad\{\mathcal{D}^2_{1/2}(M)\}\,.$$

For $\lambda,\mu\neq 0,\frac{1}{2},1$, there exists a unique (up to a constant) intertwining operator

$$\mathcal{L}^2_{\lambda,\mu}:\mathcal{D}^2_\lambda(M)\to\mathcal{D}^2_\mu(M)$$

This has been proven in [6] within the class of local maps. It has been shown in [13] that each $\mathrm{Vect}(M)$-equivariant map from $\mathcal{D}^2_\lambda(M)\to\mathcal{D}^2_\mu(M)$ is local.

Let us express $\mathcal{L}^2_{\lambda,\mu}$ in terms of the $\mathrm{sl}_{n+1}$-equivariant symbol (4.6). It follows from Corollary 2.6, that the map $\sigma_\mu\circ\mathcal{L}^2_{\lambda,\mu}\circ\sigma_\lambda^{-1}$ is diagonal.

**Proposition 7.6** *The $\mathrm{Vect}(M)$-module isomorphism $\sigma_\mu\circ\mathcal{L}^2_{\lambda,\mu}\circ\sigma_\lambda^{-1}$ is of the form :*

$$(P_2,\ P_1,\ P_0)\mapsto\left(P_2,\ \frac{2\mu-1}{2\lambda-1}\,P_1,\ \frac{\mu(\mu-1)}{\lambda(\lambda-1)}\,P_0\right)\,. \tag{7.4}$$

*Proof.* This formula follows from the explicit expression for the Vect($M$)-action (6.2) in the case of second order operators. Straightforward computations (cf. Section 5.5) give

$$\sigma_\mu \circ \mathcal{L}^2_{\lambda,\mu} \circ \sigma_\lambda^{-1} = L_X + (2\lambda - 1)\widetilde{\gamma}_1(X) + \lambda(\lambda - 1)\widetilde{\gamma}_2(X)$$

where the maps $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ do not depend on $\lambda$; $\widetilde{\gamma}_2$ is defined by (6.12) according to Proposition 6.3, and $\widetilde{\gamma}_1$ is defined by (6.7,6.8). As in the proof of Theorem 7.3, one notes that the map defined by (7.4) is intrinsic. ■

Let us now give a coordinate free expression for the map $\mathcal{L}^2_{\lambda,\mu}$.

Every second order differential operator can be written as a sum of :
(i) a zero-order operator $\phi \mapsto f\phi$ (multiplication by a function),
(ii) a first order operator $L^\lambda_X$ (Lie derivative),
(iii) a symmetric expression $[L^\lambda_X, L^\lambda_Y]_+ = L^\lambda_X \circ L^\lambda_Y + L^\lambda_Y \circ L^\lambda_X$,
where $f \in C^\infty(M), X, Y, Z \in \mathrm{Vect}(M)$.

**Proposition 7.7** *One has*

$$\begin{aligned} \mathcal{L}^2_{\lambda,\mu}\Big([L^\lambda_X, L^\lambda_Y]_+\Big) &= [L^\mu_X, L^\mu_Y]_+ \\ \mathcal{L}^2_{\lambda,\mu}(L^\lambda_Z) &= \frac{2\mu - 1}{2\lambda - 1} L^\mu_Z \\ \mathcal{L}^2_{\lambda,\mu}(f) &= \frac{\mu(\mu-1)}{\lambda(\lambda-1)} f \end{aligned}$$

*Proof.* Straightforward ■

**Remarks**.

(a) The expression for $\mathcal{L}^2_{\lambda,\mu}$ in terms of Lie derivatives is intrinsic, but it is a nontrivial fact that it does not depend on the choice of $X, Y$ and $f$ representing the same differential operator. The expression for $\mathcal{L}^2_{\lambda,\mu}$ in terms of symbols is well-defined locally, but it is a nontrivial fact that it is invariant with respect to coordinate changes. The two facts are corollaries of the third one: the two formulæ represent the same map.

(b) The explicit formula for $\mathcal{L}^2_{\lambda,\mu}$ in terms of coefficients of differential operators was obtained in [6].

## 7.6 Modules of (pseudo)differential operators in the one-dimensional case

We now study the space of (pseudo)differential operators on $S^1$ (or on $\mathbb{R}$) :

$$A = \sum_{i=0}^{\infty} a_{k-i} \left(\frac{d}{dx}\right)^{k-i}, \tag{7.5}$$

where $k \in \mathbb{R}$.

The group Diff($S^1$) and the Lie algebra Vect($S^1$) act on the space of pseudodifferential operators in the same way as on the space of differential operators. Denote $\Psi\mathcal{D}^k_\lambda$ the Vect($S^1$)-modules of the operators (7.5) acting on $\mathcal{F}_\lambda$.

**Definition 7.8** *The bilinear operations $J_m : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+m}$ given by :*

$$J_m(\phi,\psi) = \sum_{i+j=m} (-1)^i m! \binom{2\lambda+m-1}{i} \binom{2\mu+m-1}{j} \phi^{(i)}\psi^{(j)} \tag{7.6}$$

*is called the transvectants.*

It is a classical fact that the transvectants are $Sl_{n+1}$-equivariant. Moreover, for generic $\lambda,\mu$ the operations (7.6) are characterized by this property.

Put $\lambda = -1$ and $\phi = X$ a vector field on $S^1$, the transvectants (7.6) define the linear maps

$$\bar{\gamma}_m : \mathrm{Vect}(S^1) \to \mathrm{Hom}(\mathcal{F}_\mu, \mathcal{F}_{\mu+m-1}) \tag{7.7}$$

such that $\bar{\gamma}_m(X,\psi) := J_m(X,\psi)$.

**Lemma 7.9** *The* $\mathrm{Vect}(S^1)$*-action on* $\Psi\mathcal{D}^k_\lambda$ *is of the form*

$$\sigma_\lambda \circ \mathcal{L}_X \circ \sigma_\lambda^{-1} = L_X + \sum_{i=s+2}^{k} t_i^{i-s}(\lambda)\, \bar{\gamma}_{i-s+1}(X) \tag{7.8}$$

*where $t_i^{i-s}(\lambda)$ are some polynomials and $\bar{\gamma}_m$ are the operations (7.7).*

*Proof.* The formula (7.8) is a particular case of (6.2). If $n = 1$, then $\gamma_1 \equiv 0$ (cf. Proposition 6.3) and $\gamma_m$ are proportional to $\bar{\gamma}_m$ since the transvectants are unique $\mathrm{sl}_2$-equivariant bilinear maps. ■

We will study the quotient-modules $\Psi\mathcal{D}^k_\lambda/\Psi\mathcal{D}^{k-\ell}_\lambda$. If $k \in \mathbb{N}, \ell = k+1$, it is just the module of differential operators on $S^1$.

## 7.7 Classification of the modules $\Psi\mathcal{D}^k_\lambda/\Psi\mathcal{D}^{k-\ell}_\lambda$

The classification of $\mathrm{Vect}(S^1)$-modules $\Psi\mathcal{D}^k_\lambda/\Psi\mathcal{D}^{k-\ell}_\lambda$ follows from the formula (7.8). As in the multi-dimensional case, zeroes of the polynomials $t^j_{k-\ell}(\lambda)$ corresponds to exceptional modules.

Let us formulate the result for generic values of $k$.

**Proposition 7.10** *If $k \neq 0, 1/2, 1, 3/2, \ldots$, then the* $\mathrm{Vect}(S^1)$*-modules $\Psi\mathcal{D}^k_\lambda/\Psi\mathcal{D}^{k-\ell}_\lambda$ and $\Psi\mathcal{D}^k_\mu/\Psi\mathcal{D}^{k-\ell}_\mu$, are isomorphic in the following cases:*
*(i) $\ell \geq 2$;*
*(ii) $\ell = 3$, if $t^2_k(\lambda), t^2_k(\mu) \neq 0$;*
*(iii) $\ell = 4$, if $\lambda,\mu$ are not root of the polynomials $t^2_k, t^2_{k-1}, t^3_k$;*
*(iii) $\ell \geq 4$, if and only if $\lambda + \mu = 1$.*

The proof is analogous to the proofs of Theorems 7.3 and 7.5. ■

### 7.8 Relation to the Bernoulli polynomials

Let us give the explicit formulæ for the polynomials $t_k^2, t_{k-1}^2$ and $t_k^3$ :

$$
\begin{aligned}
t_k^2(\lambda) &= \frac{k(k-1)}{2k-1}\left(\lambda^2 - \lambda - \frac{(k+1)(k-2)}{12}\right) \\
t_k^3(\lambda) &= \frac{k}{6}\lambda(2\lambda-1)(\lambda-1)
\end{aligned}
$$

They evoke a possible relationship between the polynomials $t_k^j(\lambda)$ and the well-known *Bernoulli polynomials*. Indeed,

$$
t_k^2(\lambda) = \frac{k(k-1)}{2k-1}\left(B_2(\lambda) - \frac{k(k-1)}{12}\right), \quad t_k^3(\lambda) = \frac{k}{12}B_3(\lambda)
$$

where $B_s$ is the Bernoulli polynomial of degree $s$, e.g.:

$$
\begin{aligned}
&B_0(x) = 1, \quad B_1(x) = x - 1/2, \quad B_2(x) = x^2 - x + 1/6, \\
&B_3(x) = x^3 - 3x^2/2 + x/2, \quad B_4(x) = x^4 - 2x^3 + x^2 - 1/30, \\
&B_5(x) = x^5 - 5x^4/2 + 5x^3/3 - x/6.
\end{aligned}
$$

The next examples are:

$$
\begin{aligned}
t_k^4(\lambda) = \frac{k(k-1)(k-2)}{2(2k-3)(2k-5)}&\left(B_4(\lambda) + \frac{2k^2-6k+3}{24}B_2(\lambda)\right. \\
&\left. - \frac{3k^4+18k^3-35k^2+8k+2}{480}\right), \\
t_k^5(\lambda) = \frac{k(k-1)}{15(2k-7)}&\left(B_5(\lambda) + \frac{5(k-1)(k-3)}{24}B_3(\lambda)\right)
\end{aligned}
$$

**Proposition 7.11** *Polynomials $t_k^{2j}(\lambda)$ are combinations of $B_{2s}$ with $s = 0, 1, \ldots, j$ and polynomials $t_k^{2j+1}(\lambda)$ are combinations of $B_{2s+1}$ with $s = 0, 1, \ldots, j$*

*Proof.* This statement is a simple corollary of the isomorphism $\mathcal{D}_\lambda \cong \mathcal{D}_{1-\lambda}$. Indeed, under the involution $\lambda' = 1/2 - \lambda$, one has $B_s(\lambda') = (-1)^s B_s(\lambda)$. ∎

**Remark: the duality**. There exists a nondegenerate natural pairing of $\Psi\mathcal{D}^k/\Psi\mathcal{D}^\ell$ and $\Psi\mathcal{D}^{-\ell-2}/\Psi\mathcal{D}^{-k-2}$. It is given by the so-called *Adler trace* [1]: if $A \in \Psi\mathcal{D}^k$, where $k \in \mathbf{Z}$, then

$$
\mathrm{tr}(A) = \int_{S^1} a_1(x)dx.
$$

Let now $A \in \Psi\mathcal{D}^k/\Psi\mathcal{D}^\ell$ and $B \in \Psi\mathcal{D}^{-\ell-2}/\Psi\mathcal{D}^{-k-2}$. Put $(A,B) := tr(\widetilde{A}\widetilde{B})$, where $\widetilde{A} \in \Psi\mathcal{D}^k, \widetilde{B} \in \Psi\mathcal{D}^{-\ell-2}$ are arbitrary lifts of $A$ and $B$.

Adler's trace is equivariant with respect to the $\mathrm{Vect}(S^1)$-action. This means that the pairing $(\ ,\ )$ is well-defined on $\mathrm{Vect}(S^1)$-modules. Indeed, $([L_X^\lambda, A], B) + (A, [L_X^\lambda, B]) = 0$ for every $X \in \mathrm{Vect}(S^1)$ (see [4] for the details and interesting properties of the transvectants).

# 8 Conclusion

## 8.1 Generalization for the locally projective manifolds

A projective structure on a manifold $M$ is defined by an atlas with linear-fractional coordinate changes.

More precisely, a covering $(U_i)$ with a family of local diffeomorphisms $\phi_i : U_i \to \mathbb{RP}^n$ is called a projective atlas if the local transformations $\phi_j \circ \phi_i^{-1} : \mathbb{RP}^n \to \mathbb{RP}^n$ are projective (i.e. are given by the action of the group $\mathrm{SL}(n+1,\mathbb{R})$ on $\mathbb{RP}^n$).

A projective structure defines locally on $M$ an action of the Lie group $\mathrm{SL}(n+1,\mathbb{R})$ by *linear-fractional transformations* and a (locally defined) action of the Lie algebra $\mathrm{sl}(n+1,\mathbb{R})$ generated by vector fields (3.1), for every system of local coordinates of a projective atlas. This action is stable with respect to linear-fractional transformations (the space of vector fields (3.1) is well-defined globally on $\mathbb{RP}^n$).

One has the following simple corollary of Theorem 4.1.

**Corollary 8.1** *Given a manifold $M$ endowed with a projective structure, the symbol map $\sigma_\lambda$, given in local coordinates of an arbitrary projective atlas by the formulæ (4.5,4.6), is well defined globally on $M$.*

## 8.2 $\mathrm{sl}(n+1,\mathbb{R})$-equivariant star-products on $T^*M$

Let us show that for every $\lambda$ the $\mathrm{sl}(n+1,\mathbb{R})$-equivariant quantization map $\sigma_\lambda$ defines a star-product on $T^*M$ thus obtaining a 1-parameter family of $\mathrm{sl}(n+1,\mathbb{R})$-equivariant star-products.

Given a quantization map $\sigma^{-1} : \mathrm{Pol}(T^*M) \to \mathcal{D}(M)$, let us introduce a new parameter $\hbar$. For a homogeneous polynomial $P$ of degree $k$ we set

$$Q_\hbar(P) = \hbar^k \sigma^{-1}(P).$$

and we define a new associative but non-commutative multiplication on $\mathrm{Pol}(T^*M)$ by

$$F \star_\hbar G := Q_\hbar^{-1}(Q_\hbar(F) \cdot Q_\hbar(G)). \tag{8.1}$$

The corresponding algebra is isomorphic to the associative algebra of differential operators on $M$.

The result of the operation (8.1) is a formal series in $\hbar$. It has the following form:

$$F \star_\hbar G = FG + \sum_{k \geq 1} \hbar^k C_k(F,G),$$

where the higher order terms $C_k(F,G)$ are some differential operators.

Recall that such an operation is called a star-product if the skew symmetric part of $C_1(F,G)$ is the standard Poisson bracket on $\mathrm{Pol}(T^*M)$.

An elementary calculation shows that the associative operation corresponding to the quantization map (4.14) has this property.

### 8.3 Modules of differential operators and cohomology of $\mathrm{Vect}(M)$

According to the formula (2.9), the $\mathrm{Vect}(M)$-module of differential operators $\mathcal{D}_\lambda(M)$, for every $\lambda$, can be naturally viewed as a deformation of the module $\mathrm{Pol}(T^*\mathbb{R}^n)$ of symmetric contravariant tensor fields on $M$. This leads to an interesting link with the cohomology of $\mathrm{Vect}(M)$ with the operator coefficients, namely, with values in the space $\mathrm{Hom}(\mathcal{S}^k, \mathcal{S}^{k-\ell})$. Notice that this is the next natural case comparing with the Gel'fand-Fuchs cohomology of $\mathrm{Vect}(M)$ with coefficients in the modules of tensor fields on $M$.

Examples of the $\mathrm{Vect}(M)$-cohomology classes with values in $\mathrm{Hom}(\mathcal{S}^k, \mathcal{S}^{k-\ell})$ are given by the 1-cocycles $\gamma_1^\lambda$ and $\gamma_2^{1/2}$ (cf. Section 6.1). We will study this cohomology of $\mathrm{Vect}(M)$ in a subsequent article.

The restriction of the modules $\mathcal{D}_\lambda$ to $\mathrm{sl}(n+1,\mathbb{R})$ leads to the $\mathrm{sl}(n+1,\mathbb{R})$-cohomology (with the same coefficients). The complete answer in this case was obtained in [12].

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